

# Singularity-free model of electric charge in physical vacuum: Non-zero spatial extent and mass generation

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We propose a model of a spinless electrical charge as a self-consistent field configuration of the electromagnetic (EM) field interacting with a physical vacuum effectively described by the logarithmic quantum Bose liquid. We show that, in contrast to the EM field propagating in a trivial vacuum, a regular solution does exist, and both its mass and spatial extent emerge naturally from dynamics. It is demonstrated that the charge and energy density distribution acquire Gaussian-like form. The solution in the logarithmic model is stable and energetically favourable, unlike that obtained in a model with a quartic (Higgs-like) potential.

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## 1. INTRODUCTION

Two of the oldest actual problems in fundamental particle physics relate to finite self-energy and the possible extendedness of electrically charged elementary particles. This research direction is complicated by the fact that no experimental evidence of either the internal structure or spatial extent of, e.g., the electron has been found down to  $10^{-16}$  cm. The mere postulate that a certain amount of matter with mass, charge and spin can be located inside a set of zero spatial measure looks implausible to a physicist's mind. This assumption, however, might be one of the reasons why unphysical divergences appear in quantum field theory (QFT). This difficulty already arises at the classical level: according to the standard theory of electromagnetism, the electrical field of a point charge in completely empty space (trivial vacuum) is described by the inverse square law, therefore the energy density of the electrical field integrated over the whole space turns out to be infinite. As a result, the total mass-energy of the point charge, together with its field, becomes infinite, therefore, such a system would be impossible to move. At the quantum level, this problem manifests itself in ultraviolet divergences appearing in loop diagrams. In some theories, these divergences can be removed by means of regularization and renormalization procedures. This can be very useful for doing specific computations, but does not shed much light upon the essence of the problem [1–5]. Theoretical attempts towards better understanding should not, however, be abandoned.

Historically the first effort to address this problem was

probably a model which described the electron as a ball with spatially distributed electrical charge. That model conflicted with relativity, however, because the ball was assumed to be absolutely rigid. The description of spin was also not clear. Similar difficulties were found in models proposed by Abraham and Lorentz [6–8], although work in that direction continues [9, 10]. Dirac's shell model of the extended electron, together with its subsequent modifications and variations [11–14], provides another notable research direction. Yet another model of an extended electrical charge was the Einstein's wormhole approach. In his model of an electron the electrical flux lines enter one side of the wormhole and exit from another, resulting in the front side looking like a negative charge and the rear like a positive charge. The wormhole models were criticized by Wheeler for issues of stability, non-quantized charge, wrong mass-charge ratio and spin [15]. Numerous attempts towards finding regular particle-like solutions were made in conventional and nonlinear electrodynamics [16], both with and without engaging general relativity [17–28]. Another interesting approach is wave-corpuscle mechanics (for a review see [29]). The general mathematical formalism used therein formally resembles the one used in this paper. The important difference, however, is about underlying physics: the origin and explicit form of their nonlinear self-interacting wave term  $G(|\psi|^2)$  are not specified on physical grounds and a fully satisfactory particle-like solution is not given.

A popular approach to the classical electron model was made using the Einstein-Dirac [18] and Kerr-Newman (KN) solutions [30–42] (an extensive bibliography can be found in [43]). While the original KN solution does have the correct gyromagnetic ratio, it also contains a naked singularity and thus requires an additional mechanism to circumvent the regularization problem; the story is far from being complete yet.

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Intuitively, from a fundamental theory one would expect that spatial extent is not *ab initio* built-in, but naturally emerges from dynamics. In this paper we propose a model of a charged particle whose spatial extent, observable charge and mass emerge as a result of the interaction of the EM field with the physical vacuum. For simplicity we neglect internal degrees of freedom, such as spin, isospin, etc., so that we can assume spherical symmetry where possible. The resulting solution describes a charged object which does not have a boundary in a classical sense; its stability is supported not by surface tension but by nonlinear quantum effects in the bulk. This makes our model more realistic from the quantum-mechanical point of view, since the actual observability of a definite boundary with smooth surface would be as contradictory to the quantum uncertainty principle as the notion of a smooth trajectory or worldline in the quantum realm. This can be shown by performing a simple *Gedankenexperiment*: making an extended object with a definite boundary propagate through space and measuring the velocity and position uncertainties on its surface. In turn, it means that the surface tension is a well-defined notion only in the classical limit, but for more fundamental purposes it must be used with utmost care.

The structure of the paper is as follows. The phenomenological approach to physical vacuum is described in the next section, the main equations of the model can be found in Section 3, the regular solution and its properties are analyzed, both analytically and numerically, in Section 4. A comparison between our solution and its Higgs-type (quartic) counterpart is done in Section 5, and conclusions are drawn in Section 6.

## 2. PHYSICAL VACUUM

As mentioned above, the Coulomb divergence problem essentially means that one cannot find regular particle-like solutions of the Maxwell field in empty space, not even in general relativity [44]. From a quantum physicist's point of view, however, this problem is not as severe as it looks to a non-quantum theorist, because the notion of absolutely empty space (or “mathematical vacuum”) cannot be realized in nature anyway. This is because the existence of such space seriously contradicts quantum-mechanical laws. According to the latter, the genuine (physical) vacuum must be a non-trivial quantum medium which acts as a non-removable background and affects particles propagating through [1, 45].

At this time, no commonly accepted theory of physical vacuum exists. The amount of experimental and observational data is still too far from conclusively identifying a single model. One of the candidate theories lies within the framework of the superfluid vacuum approach [46–51]. This theory is based on the idea [52–54] that a physical vacuum can be viewed as some sort of background superfluid condensate described by the logarithmic wave equation (the latter was studied previously on grounds

of the dilatation covariance [55] or separability [56–58]). It was shown that small fluctuations of the logarithmic condensate obey the Lorentz symmetry and can be interpreted not only as the relativistic particle-like states but also as the gravitational ones [54] depending on a type of mode. In this approach, therefore, the Lorentz symmetry is not an exact symmetry of nature, but rather pertains to small fluctuations of the physical vacuum, and thus gets deformed at high energies and/or momenta.

As long as the superfluid vacuum approach must be fully consistent and applicable to reality, one would expect that if the empty space is replaced by the logarithmic vacuum condensate then the behaviour of the conventional Maxwell field becomes regular in the presence of particle-like solutions. This behaviour is going to be the main subject of the current study. One would also expect that the spatial extent mentioned above must appear naturally in the approach. This has already been shown at the non-relativistic level in [59], so in the current study we will investigate the relativistic case.

The effective low-energy Lagrangian for the Maxwell field interacting with small fluctuations of the physical vacuum was proposed in [54], along with a mass generation mechanism which was analogous to the Higgs one. In this paper we propose a different mass generation mechanism which uses the same Lagrangian, except that the scalar potential is assumed to be “upended”. As will be shown below, the latter solves the issue of a wrong sign in the quadratic (mass) term of the potential at energies above the symmetry-breaking scale - when symmetry is unbroken and false vacuum is stable. In the case of three spatial dimensions the action is proportional to  $\int d^4x \mathcal{L}$  where, adopting the natural units  $c = \hbar = \epsilon_0 = 1$  and metric signature  $(+ - - -)$ , we assume

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\check{\alpha}}{2}|D_\mu\Psi|^2 - V(|\Psi|^2), \quad (1)$$

where  $D_\mu = \partial_\mu + igA_\mu$ ,  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , and the vacuum-induced field potential is defined as (up to an additive constant)

$$V(|\Psi|^2) = -\check{\beta}^{-1} \{ |\Psi|^2 [\ln(\check{\alpha}^3|\Psi|^2) - 1] + \check{\alpha}^{-3} \}, \quad (2)$$

where  $\check{\alpha}$  and  $\check{\beta}$  are parameters of dimensionality length in adopted units. In the underlying theory of superfluid vacuum the parameter  $\check{\beta}$  is related to the quantum (non-thermal) temperature which is conjugated to the Everett-Hirschmann information entropy [59] whereas  $\check{\alpha}$  can be related to the characteristic inhomogeneity scale of the superfluid [60]. The potential (2) is regular in the origin - while the logarithm itself diverges there, the factor  $|\Psi|^2$  recovers regularity. There is a local maximum at  $|\Psi|_{\max} = \check{\alpha}^{-3/2}$ , i.e., it always has the (upside-down) Mexican-hat shape if plotted as a function of  $\Psi$ , see Fig. 1. In what follows we call this potential *logarithmic* - due to the property  $dV/d|\Psi|^2 \propto \ln(\check{\alpha}^3|\Psi|^2)$  which yields the logarithmic term in the corresponding field equation.

We emphasize that, according to this approach, the Lagrangian (1) is an approximate one, thus it is not valid

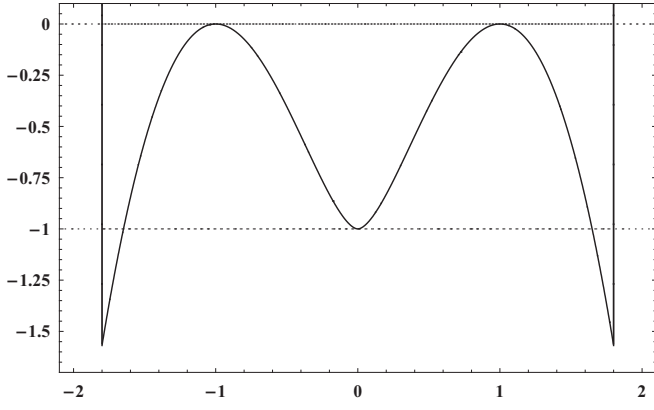


FIG. 1: The field-theoretical potential (2) (in units of  $\check{\beta}\check{a}^3$ ) versus  $\text{Re}(\Psi)$  (in units of  $\check{a}^{-3/2}$ ). Vertical lines conditionally represent inequality (3). In an approximation when the symmetry-breaking energy scale is much less than the vacuum one ( $\gtrsim 10$  TeV) these walls can be assumed infinite.

for arbitrarily large (or short) scales of energy (or length) which means that the  $\Psi$  field cannot take arbitrarily large values:

$$|\Psi| \leq |\Psi_c| < \infty, \quad (3)$$

where  $|\Psi_c| = \lim_{E \rightarrow E_0} |\Psi|$  is some limit value corresponding to the cutoff energy scale  $E_0$  which is also the characteristic energy scale of the vacuum. In the effective theory the appearance of upper bound for  $|\Psi|$  will be shown in Section 4.2 below, whereas in a full theory it could result from, for instance, the normalization condition for  $\Psi$ , similarly to the one in the theory of Bose-Einstein condensation. The constraint (3) also means that the potential (2) does not have to be bounded from below, as in a standard relativistic QFT. Alternatively, one can take the potential (2) with an opposite sign, thus making it bounded from below at positive  $\check{\beta}$ , but treat  $\Psi$  as a phantom field.

Further, performing the rescaling

$$\psi = \sqrt{\check{a}}\Psi, \quad \beta = \check{a}\check{\beta}, \quad a = \check{a}^{2/3}, \quad (4)$$

we can rewrite (1) and (2) in a more regular form:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}|D_\mu\psi|^2 - V(\psi), \quad (5)$$

$$V(\psi) = -\beta^{-1} \left\{ |\psi|^2 [\ln(a^3|\psi|^2) - 1] + \frac{1}{a^3} \right\}. \quad (6)$$

Note that by expanding the potential in the vicinity of  $|\psi|^2 = \varepsilon/a^3$ ,  $\varepsilon$  being a non-negative dimensionless number, one arrives at the following perturbative expression (up to an additive constant):

$$V(\psi) \approx \frac{\lambda_{\text{eff}}}{4!}|\psi|^4 + \frac{1}{2}m_b^2|\psi|^2 + \mathcal{O}((|\psi|^2 - \varepsilon/a^3)^3), \quad (7)$$

where  $\lambda_{\text{eff}} = -12a^3/\varepsilon\beta$  is the effective quartic coupling and  $m_b = \sqrt{2(1 - \ln \varepsilon)}/\beta$ . If the radicand of  $m_b$  is non-negative then  $m_b$  can be interpreted as the mass of an effective scalar particle (before the symmetry breaking). Indeed, one can always quantize the approximate model by analogy with a quartic (hence renormalizable) scalar QFT in the vicinity of a non-trivial vacuum represented by the ground-state solution of the original model (6) which will be discussed in the following sections. To date, a number of different quantization approaches have been developed for such cases - see, e.g., works [61–63] and references therein.

We have now expressed the physical parameters of the scalar sector, such as mass and coupling, in terms of the primary parameters of our theory. If the value of  $\varepsilon$  is close to one (or, at least, less than the base of natural logarithm) then it is indeed important that the potential (6) has the upside-down Mexican hat shape (cf.  $\check{a} > 0$  and  $\check{\beta} > 0$ ) otherwise the quadratic term would appear with a wrong sign. Also the effective quartic coupling turns out to be negative in this case which is a remarkable difference from the standard Higgs potential and thus it can serve for experimental testing. In any case, the interpretation based on (7) is only approximate (for instance, such series expansion does not converge to (6) for very small  $|\psi|$ ), therefore, in what follows we will be working with the exact expression for  $V$ .

### 3. FIELD EQUATIONS

The field equations corresponding to the Lagrangian (5) are given by

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad (8)$$

$$D_\mu D^\mu \psi + \frac{\partial V}{\partial \psi^*} = [D_\mu D^\mu + \beta^{-1} \ln(a^3|\psi|^2)] \psi = 0, \quad (9)$$

where the current  $j^\nu$  is defined as

$$j^\nu = ig [(D^\mu \psi)^* \psi - \psi^* (D^\mu \psi)]. \quad (10)$$

We look for a solution in the electrostatic form

$$A_\mu = [\phi(r), \vec{0}], \quad \psi(t, r) = e^{-iEt} \psi(r), \quad (11)$$

with  $E$  being a real-valued constant. Then the equations of motion become simply

$$\psi'' + \frac{2}{r}\psi' = -(E - g\phi)^2 \psi - \beta^{-1} \psi \ln(a^3\psi^2), \quad (12)$$

$$\phi'' + \frac{2}{r}\phi' = -2g(E - g\phi)\psi^2, \quad (13)$$

where primes indicate derivatives with respect to  $r$ . Introducing the quantities

$$\begin{aligned} x &= \frac{r}{\sqrt{\beta}}, \quad \tilde{\psi} = a^{3/2}\psi = \check{a}^{3/2}\Psi, \quad \tilde{\phi} = \\ &= -g\phi\sqrt{\beta} = -g\phi\sqrt{\check{a}\check{\beta}}, \end{aligned} \quad (14)$$

$$\tilde{E} = E\sqrt{\beta} = E\sqrt{\check{a}\check{\beta}}, \quad \tilde{g} = g\sqrt{\frac{\beta}{a^3}} = g\sqrt{\frac{\check{\beta}}{\check{a}}}, \quad (15)$$

we obtain

$$\tilde{\psi}'' + \frac{2}{x}\tilde{\psi}' = -\left(\tilde{E} + \tilde{\phi}\right)^2 \tilde{\psi} - \tilde{\psi} \ln(\tilde{\psi}^2), \quad (16)$$

$$\tilde{\phi}'' + \frac{2}{x}\tilde{\phi}' = 2\tilde{g}^2 \left(\tilde{E} + \tilde{\phi}\right) \tilde{\psi}^2, \quad (17)$$

where primes indicate derivatives with respect to  $x$  when applied to tilded quantities. One can see that equations depend on only one parameter  $\tilde{g}$ , whereas  $\tilde{E} = \tilde{E}(\tilde{g})$  can be treated as an eigenvalue at a given  $\tilde{g}$ . The full theory of physical vacuum would provide the value of  $\tilde{E}$  (or, equivalently,  $\tilde{\psi}(0)$ , see the analytical solution section below) as a function of the primary parameters (from, e.g.,

some sort of normalization condition, cf. [59]) but in the approximate theory  $\tilde{E}$  stays a free parameter which can be fixed only from external considerations.

Further, for a given solution, the energy density  $\epsilon$  is defined as

$$\epsilon = \frac{1}{2}D_0\psi^*D^0\psi + \frac{1}{2}D_i\psi^*D^i\psi + V(\psi) + \frac{1}{2}|\vec{\mathcal{E}}|^2, \quad (18)$$

where  $i = 1, 2, 3$  and  $\mathcal{E} = -\nabla\phi$  is the electric field strength. Substituting the ansatz (11) we obtain

$$\beta a^3 \epsilon = \frac{1}{2} \left(\tilde{E} + \tilde{\phi}\right)^2 \tilde{\psi}^2 + \frac{1}{2} \tilde{\psi}'^2 - \tilde{\psi}^2 \ln(\tilde{\psi}^2) + \frac{1}{2\tilde{g}^2} \tilde{\phi}'^2. \quad (19)$$

The total energy can be calculated as  $W = 4\pi \int_0^\infty r^2 \epsilon(r) dr$  so we obtain

$$W = \frac{4\pi\sqrt{\beta}}{a^3} \tilde{W}(\tilde{g}), \quad (20)$$

where we denoted the dimensionless total energy

$$\tilde{W}(\tilde{g}) = \frac{1}{2} \int_0^\infty \left[ \left(\tilde{E} + \tilde{\phi}\right)^2 \tilde{\psi}^2 + \tilde{\psi}'^2 - 2\tilde{\psi}^2 \ln(\tilde{\psi}^2) + \left(\frac{\tilde{\phi}'}{\tilde{g}}\right)^2 \right] x^2 dx. \quad (21)$$

For an observer in the reference frame associated with the center of mass of a localized solution the quantity  $W/c^2$  is equivalent to the rest mass of a corresponding particle.

## 4. PARTICLE-LIKE SOLUTION AND ITS PROPERTIES

### 4.1. Asymptotic behaviour

To search for the regular solution, the functions  $\tilde{\psi}(x)$  and  $\tilde{\phi}(x)$  should have the following behaviour near the origin:

$$\tilde{\psi}(x) = \tilde{\psi}_0 + \tilde{\psi}_2 \frac{x^2}{2} + \mathcal{O}(x^4), \quad (22)$$

$$\tilde{\phi}(x) = \tilde{\phi}_0 + \tilde{\phi}_2 \frac{x^2}{2} + \mathcal{O}(x^4). \quad (23)$$

When substituting this into (16) and (17) we obtain the solution

$$\tilde{\psi}_2 = -\frac{\tilde{\psi}_0}{3} \left[ \left(\tilde{E} + \tilde{\phi}_0\right)^2 + \ln \tilde{\psi}_0^2 \right], \quad (24)$$

$$\tilde{\phi}_2 = \frac{2}{3} \tilde{g}^2 \left(\tilde{E} + \tilde{\phi}_0\right) \tilde{\psi}_0^2. \quad (25)$$

Further, the asymptotic behaviour at  $x \rightarrow \infty$  is given by

$$\tilde{\psi} \rightarrow e^{\frac{1}{2}(3-\tilde{E}^2-x^2)} [1 + \mathcal{O}(\tilde{g}^2)], \quad \tilde{\phi} \rightarrow -\frac{\tilde{q}}{x}, \quad (26)$$

where  $\tilde{q}$  is some constant to be determined. It is instructive to relate the bare charge  $g$  to the observable one  $q = \tilde{q}/g$ . By integrating (13) we obtain

$$r^2 \mathcal{E} = 2g \int_0^r (E - g\phi) \psi^2 r^2 dr, \quad (27)$$

and in the limit  $r \rightarrow \infty$  we have  $r^2 \mathcal{E} \rightarrow q$  hence

$$q = 2g \int_0^\infty (E - g\phi) \psi^2 r^2 dr. \quad (28)$$

Thus, we arrive at the following relation between the bare and observable charges

$$q = \frac{2g\beta}{a^3} I(\tilde{g}) = 2\tilde{g} \sqrt{\frac{\beta}{a^3}} I(\tilde{g}), \quad (29)$$

where we denoted

$$I(\tilde{g}) = \int_0^\infty x^2 \left( \tilde{E} + \tilde{\phi} \right) \tilde{\psi}^2 dx. \quad (30)$$

Expression (26) shows us that at large distance we recover the Coulomb potential while the field  $\tilde{\psi}$  decreases exponentially. Thus, the field  $\tilde{\psi}$  is in fact unobservable, unless very short length scales are probed. From the asymptotics of the solution one can infer that the charge radius of the solution is determined by the parameter  $\beta$ :

$$\text{size} \sim \sqrt{\beta} \sim \sqrt{\check{a}\check{\beta}}, \quad (31)$$

which essentially means that the combination of parameters  $\beta = \check{a}\check{\beta}$  must have an extremely small value for the known elementary particles. For instance, if one takes the values of the classical radius  $e^2/m$  as conservative estimates, then for the electron and muon one would obtain constraints of  $\beta_{(e)} < 10^{-26} \text{ cm}^2$  and  $\beta_{(\mu)} < 10^{-32} \text{ cm}^2$ , although it is not entirely clear whether the classical radius should be analogous to the “smearing” size which is our definition of size here.

## 4.2. Approximate analytical solution

While the exact expression for a full analytical solution is unknown, it is possible to solve the system (16) and (17) using the approximation of weak EM coupling

$$\tilde{g}^2 \ll 1, \quad (32)$$

which is equivalent to  $g^2 \ll a^3/\beta$  or  $g^2 \ll \check{a}/\check{\beta}$ . The observational constraints suggest that this approximation might have a good chance to be valid for the known elementary particles - unless  $\check{a}$  turns out to be very small. On the other hand, as long as our approach is an effective one it has certain applicability conditions - and one of them is that the vacuum effects predominate the electromagnetic ones. Therefore, large values of  $\tilde{g}$  might push our approach outside its applicability range and thus the corresponding approximation is not very interesting from the physical point of view.

Thus, imposing the boundary conditions

$$\tilde{\phi}(0) < \infty, \quad \tilde{\phi}(+\infty) = 0, \quad \tilde{\psi}(+\infty) < \infty, \quad \tilde{\psi}'(0) = 0, \quad (33)$$

one obtains a solution which is regular for  $0 \leq r \leq +\infty$  (see appendix for the details of derivation):

$$\tilde{\phi} = -\frac{1}{2x} \sqrt{\pi} \tilde{g}^2 \tilde{E} e^{3-\tilde{E}^2} \text{erf}(x) + \mathcal{O}(\tilde{g}^4), \quad (34)$$

$$\tilde{\psi} = e^{\frac{1}{2}(3-\tilde{E}^2-x^2)} \left\{ 1 + \frac{1}{4} \tilde{g}^2 \tilde{E}^2 e^{3-\tilde{E}^2-x^2} \left[ 1 + \frac{\sqrt{\pi}}{2x} (2x^2 + 1) e^{x^2} \text{erf}(x) \right] \right\} + \mathcal{O}(\tilde{g}^4), \quad (35)$$

where dimensionless energy  $\tilde{E}$  can be also expressed via the boundary value

$$\tilde{E} = \sqrt{3 - \ln(\tilde{\psi}_0^2)} \left( 1 - \frac{1}{2} \tilde{g}^2 \tilde{\psi}_0^2 \right) + \mathcal{O}(\tilde{g}^4), \quad (36)$$

with the square root being defined up to a sign. Of course, this formula is valid only if the magnitude of  $\tilde{\psi}$  is bounded from above:

$$|\tilde{\psi}| \leq |\tilde{\psi}_0| \leq e^{3/2}, \quad (37)$$

which *a posteriori* affirms the condition of applicability (3), although this upper bound does not necessarily saturate the critical value there:  $|\Psi_c| \geq (e/\check{a})^{3/2}$ .

Further, one can check that the electric part indeed has the Coulomb behaviour at large  $r$ , and then the effective charge can be computed as

$$\tilde{q} = \frac{1}{2} \sqrt{\pi} \tilde{g}^2 \tilde{E} e^{3-\tilde{E}^2} + \mathcal{O}(\tilde{g}^4), \quad (38)$$

therefore, the observable charge,

$$q = \tilde{q}/g \approx \frac{1}{2a^{3/2}} \sqrt{\pi} \beta \tilde{E} e^{3-\tilde{E}^2} \tilde{g} \approx \frac{1}{2\check{a}} \sqrt{\pi} \check{\beta} \tilde{E} e^{3-\tilde{E}^2} g, \quad (39)$$



depends on the whole combination of parameters describing the interaction of the electromagnetic field with a physical vacuum. The dimensionless total energy (21) turns out to be

$$\tilde{W} = \frac{3}{16} \sqrt{\pi} e^{3-\tilde{E}^2} \left[ 2\tilde{E}^2 - 1 + \frac{\tilde{g}^2}{3\sqrt{2}} \tilde{E}^2 (12\tilde{E}^2 - 13) e^{3-\tilde{E}^2} \right] + \mathcal{O}(\tilde{g}^4), \quad (40)$$

which can also be written in terms of the observable charge  $q$  and rest mass  $W$ :

$$W \approx W_{(0)} \left[ 1 + 4\sqrt{2}\beta^{-1} a^3 q^2 e^{\tilde{E}^2-3} \frac{\tilde{E}^2-13/12}{\tilde{E}^2-1/2} \right], \quad (41)$$

where

$$W_{(0)} = \frac{3}{2} \frac{\pi^{3/2} \sqrt{\beta}}{a^3} \left( \tilde{E}^2 - \frac{1}{2} \right) e^{3-\tilde{E}^2}, \quad (42)$$

so one can see that the obtained formula does not contain any divergences.

It is also apparent that mass  $W$  does not vanish when charge is set to zero, which indicates that the theory is also capable of incorporating non-charged particles into the scheme, by taking the corresponding limit. In fact, the mass formula (41) implies that for an electrically charged particle with mass  $W$  there can exist not only an antiparticle of the same mass but also a neutral particle of related mass  $W_{(0)}$ . It is interesting that the ratio  $W/W_{(0)}$  grows exponentially with growing  $|\tilde{E}|$ , which results in two possible scenarios: (i) the mass of a neutral partner is very small (yet non-zero) as compared to the mass of a charged one: this happens if  $|\tilde{E}| \gg 1$  (or, equivalently,  $|\tilde{\psi}_0| \ll 1$ ); (ii) if  $|\tilde{E}|$  is of order one or less then both masses would be of the same order of magnitude. The possible phenomenological implications of this mechanism are discussed in the conclusion.

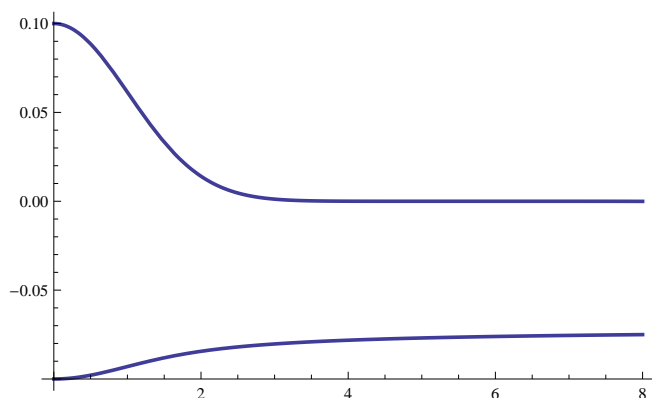


FIG. 2: Profiles of  $\tilde{\psi}(x)$  (top curve) and the electrostatic potential  $\tilde{\phi}(x)$  (bottom curve), computed for eigenvalue  $\tilde{E} = 2.8436935588$ .

### 4.3. Numerical solution and stability

While we have managed to find the approximate regular solution analytically, it is important to check that a regular solution exists for non-small  $\tilde{g}$ 's and that terms with higher-order powers of  $\tilde{g}$  will not introduce any spatial singularities. For this purpose we solve equations (16) and (17) numerically. For the computations we choose  $\tilde{g} = 1$  and the following boundary conditions

$$\tilde{\psi}(0) = 0.1, \quad \tilde{\psi}'(0) = 0, \quad \tilde{\phi}(0) = -0.1, \quad \tilde{\phi}'(0) = 0, \quad (43)$$

and  $\tilde{E}$  is treated as an eigenvalue. It should be noted that  $\tilde{\phi}(x)$  must be always taken as non-positive on the positive semi-axis of  $x$ , due to the asymptotic requirements (26). The numerical solution is presented in Fig. 2, and in Fig. 3 the corresponding profiles of the electric field  $\tilde{\mathcal{E}} = -d\tilde{\phi}/dx$  and  $x^2\tilde{\mathcal{E}}$  are given. From these one sees that the electric field is regular at the origin and asymptotically displays Coulomb behaviour. The profile of the dimensionless energy density is shown in Fig. 4.

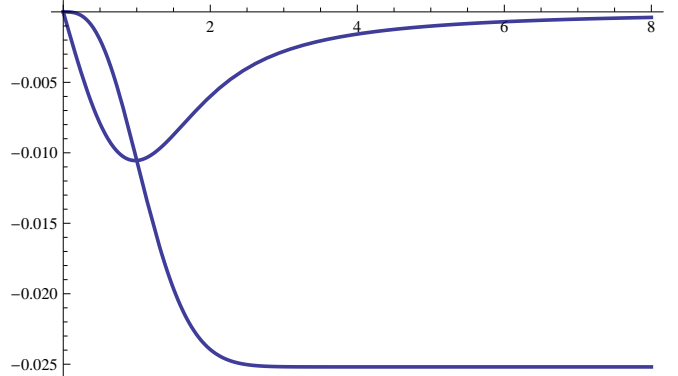


FIG. 3: Profiles of the electric field  $\tilde{\mathcal{E}}(x)$  (top) and  $x^2\tilde{\mathcal{E}}(x)$  (bottom curve).

The direct stability analysis of the solution is complicated by the fact that the perturbed electric field becomes time-dependent, which leads to the appearance of a magnetic field such that this system cannot be regarded as spherically symmetric anymore. It is, however, still possible to use the energy-based arguments as well as to investigate the behaviour of an effective Schrodinger equation potential. Let us consider the dimensionless total energy  $\tilde{W}$  given by (21), as well as the energy of the field  $\tilde{\psi}$  alone,  $\tilde{W}_\psi = \tilde{W}|_{\phi \rightarrow 0}$ , and the energy of the electric field alone,  $\tilde{W}_\phi = \tilde{W}|_{\psi \rightarrow 0}$ . Then we can define the

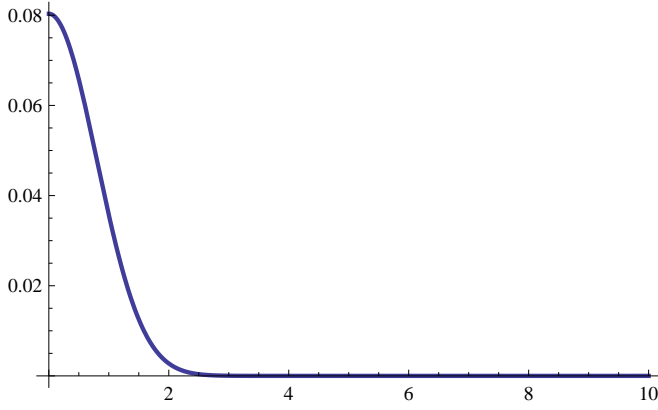


FIG. 4: Profile of dimensionless energy density  $\beta a^3 \epsilon(x)$ , using the same  $\tilde{E}$  eigenvalue as in previous figures.

dimensionless binding energy as

$$\Delta \tilde{W} = \tilde{W} - (\tilde{W}_\phi + \tilde{W}_\psi) = \frac{1}{2} \int_0^\infty \tilde{\phi} (\tilde{\phi} + 2\tilde{E}) \tilde{\psi}^2 x^2 dx, \quad (44)$$

where all the potentials are assumed to be given by our regular solution. One considers the following two possibilities. If binding energy is positive then it is necessary to add a certain amount of energy to create a regular electric charge when coupling to the  $\psi$  field. In the opposite case binding energy gets released during the process, and the energy of the whole configuration is smaller than the sum of the energies of the separate electric and scalar fields.

Evaluating the binding energy on approximate solution (34), (35), we find that it is negative-definite

$$\Delta \tilde{W} = -2^{-5/2} \sqrt{\pi} \left( \tilde{g} \tilde{E} e^{3-\tilde{E}^2} \right)^2 + \mathcal{O}(\tilde{g}^4), \quad (45)$$

which means that the creation of the regular electric charge in the logarithmic model is energetically favourable.

Another way to study the stability of the solution is to write it as a solution of the Schrödinger equation for a fictitious particle

$$-\Delta \Psi + V_{\text{eff}}(x) \Psi = \varepsilon \Psi, \quad (46)$$

where  $\Delta = d^2/dx^2$ ,  $\varepsilon = \tilde{E}^2$  and the effective potential is derived as

$$V_{\text{eff}}(x) = -2\tilde{E}\tilde{\phi}(x) - \tilde{\phi}^2(x) - \ln[\tilde{\psi}^2(x)], \quad (47)$$

where the tilded potentials are given by our regular solution. According to (26), the asymptotic behaviour of the solution implies that the effective potential  $V_{\text{eff}} \propto x^2$  at large  $x$ , (see also Fig. 5), and thus the “particle” is always localized in a finite region of  $x$ . With respect to the solution itself this means that it cannot spread or be destroyed when subjected to small perturbations.

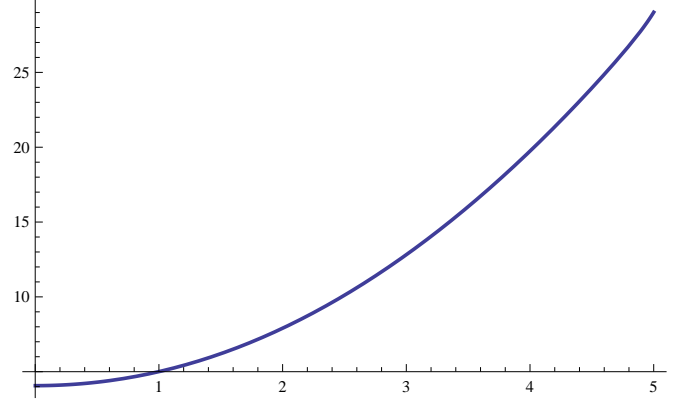


FIG. 5: Effective potential (47) versus  $x$  at  $\tilde{g}=1$ , using the same  $\tilde{E}$  eigenvalue as in previous figures.

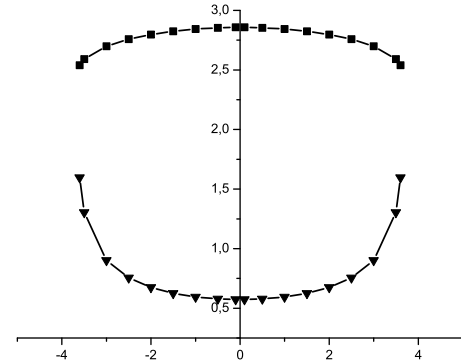


FIG. 6: The profiles of the dimensionless energy  $\tilde{W}(\tilde{g})$  ( $\nabla$ -curve) and the eigenvalues  $\tilde{E}(\tilde{g})$  ( $\blacksquare$ -curve).

## 5. LOGARITHMIC VERSUS QUARTIC POTENTIAL

One may wonder whether the singularity-free solution exists when the scalar sector of our model is controlled not by the logarithmic potential (6) but by a more orthodox one, such as the Higgs-type (quartic) potential:

$$V_H(\psi) = -\frac{\varkappa}{4} \psi^4 + \frac{m^2}{2} \psi^2. \quad (48)$$

Corresponding dimensionless equations with the ansatz (11) are

$$\tilde{\psi}'' + \frac{2}{x} \tilde{\psi}' = \left[ (\tilde{E} + \tilde{\phi})^2 - \lambda \tilde{\psi}^2 + 1 \right] \tilde{\psi}, \quad (49)$$

$$\tilde{\phi}'' + \frac{2}{x} \tilde{\phi}' = 2\tilde{g}^2 (\tilde{E} + \tilde{\phi}) \tilde{\psi}^2. \quad (50)$$

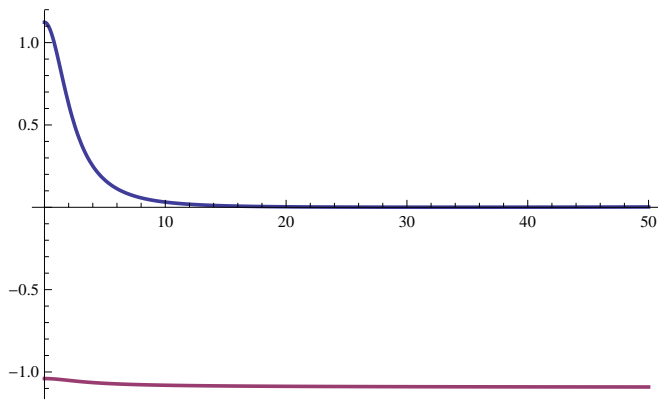


FIG. 7: The profiles of  $\tilde{\psi}(x)$  (top) and  $\tilde{\phi}(x)$  (bottom) for the quartic model. Eigenvalue  $\tilde{\psi}(0) = 1.12345$ ,  $\tilde{g} = 0.1$  and parameters  $\tilde{E} = 0.1$ ,  $\lambda = 1.0$  have been used.

where we introduced following dimensionless quantities

$$x = mr, \quad \tilde{\psi} = \frac{\psi}{\psi(0)}, \quad \tilde{E} = \frac{E}{m}, \quad \tilde{\phi} = -\frac{g\phi}{m}, \quad \lambda = \frac{\psi(0)^2 \varkappa}{m^2}. \quad (51)$$

The boundary conditions are

$$\tilde{\psi}'(0) = 0; \quad \tilde{\phi}(0) = -1.04; \quad \tilde{\phi}'(0) = 0. \quad (52)$$

As in the logarithmic model, the regular solution exists only if the potential (48) opens down, i.e., when  $\varkappa > 0$ . The profiles of  $\tilde{\psi}(x)$  and  $\tilde{\phi}(x)$  are presented in Fig. 7. For technical reasons in this case an eigenvalue is  $\tilde{\psi}(0)$  not  $\tilde{E}$ . In Fig. 8 the profile of the electric field  $\mathcal{E}$  is shown. In order to show that the electric field asymptotically has Coulomb behaviour we also present the profile  $x^2\mathcal{E}(x)$  in Fig. 8. From these figures one can see that the qualitative behaviour of the potential  $\phi(x)$  and the electric field  $\mathcal{E}(x)$  are the same as for the logarithmic potential.

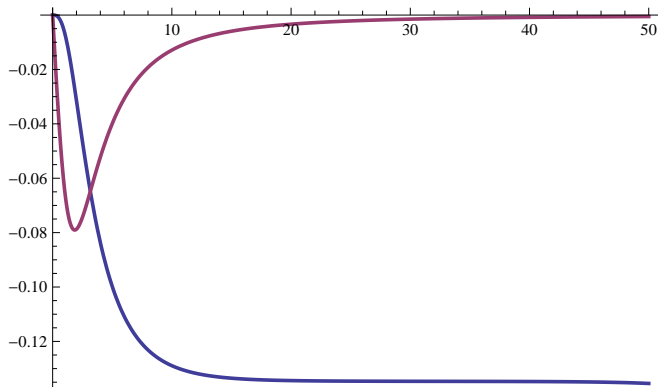


FIG. 8: Electric field versus  $x$  for the quartic model. The top curve is  $10\mathcal{E}(x)$ , the bottom one is  $x^2\mathcal{E}(x)$ .

The asymptotic behaviour for the functions  $\psi$  and  $\phi$

at large  $x$  is given by

$$\tilde{\psi}(x) \rightarrow \psi_\infty \frac{e^{-x\sqrt{\tilde{E}+1}}}{x^2}, \quad (53)$$

$$\tilde{\phi}(x) \rightarrow -\frac{\tilde{q}}{x}, \quad (54)$$

where  $\psi_\infty$  is some constant. This still looks very similar to what we had earlier in the logarithmic case, however if we study the stability of this solution then differences do arise. At first, if one computes the binding energy similarly to (44) then it turns out to be positive, therefore, the creation of the regular electric charge by coupling electrical field to the quartic scalar one is energetically unfavourable. Further, if one computes the fictitious-particle potential for this solution, cf. (47),

$$V_{\text{eff}}(x) = -2\tilde{E}\tilde{\phi}(x) - \tilde{\phi}^2(x) - \lambda\tilde{\psi}^2(x) + 1, \quad (55)$$

then one finds that it approaches a constant at large  $x$  (see also Fig. 9). The “particle” is not, therefore, necessarily localized in a finite region of  $x$ . With respect to the solution itself, this means that the latter can spread or become unstable against small perturbations.

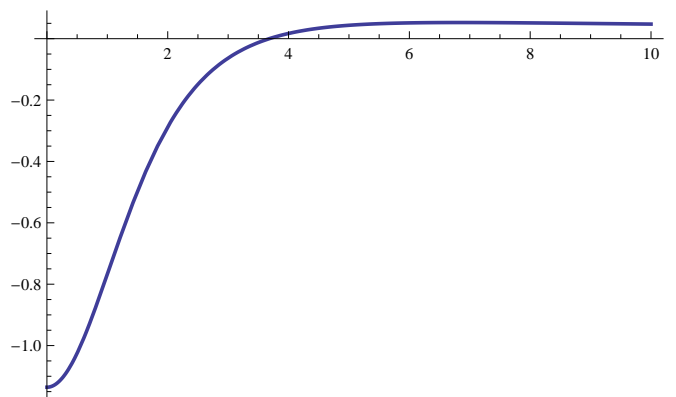


FIG. 9: Effective potential (55) versus  $x$  at  $\tilde{g} = 1$  for the quartic model.

## 6. CONCLUSION

The classical model of a spinless electrical charge is described as a self-consistent configuration of the EM field interacting with fluctuations of a nontrivial physical vacuum effectively represented by the logarithmic Bose-Einstein condensate. We have shown that a regular solution does exist - as opposed to the case of the EM field propagating in absolutely empty space. In this regard we recall the state of affairs in quantum mechanics: the Dirac/Schrödinger equation without any external potential has the de Broglie wave solution; the Dirac/Schrödinger equation with the external electrostatic field yields the regular wave functions (the hydrogen atom being an example), but the Dirac/Schrödinger



equation coupled to Maxwell equations does not lead to a regular stationary solution. The reason is that the Dirac/Schrödinger equation *ab initio* describes a point-like particle which might be a good approximation for long-wavelength measurements, but in higher-energy and shorter-length regimes this approximation eventually becomes too crude, since it neglects internal structure and non-zero spatial extent. Among other things, this leads to the densities of energy and charge becoming infinite at the particle's position. Here we have shown that by introducing an additional player on scene, the physical vacuum condensate, one can obtain a regular solution, thus endowing particles with internal structure and spatial extent. The solution turns out to be stable and energetically favourable. Using its features, some observational constraints for the parameters of the theory have been derived. We also specified the conditions under which our model can be (approximately) interpreted in terms of a scalar particle and those under which it cannot.

Further, we have established, both numerically and analytically, that the mass and spatial extent of a charged particle emerge due to interaction of the EM field with vacuum. It has been demonstrated that the average charge radius becomes non-zero, and the charge density acquires a Gaussian-like form. Looking at the form of the analytical solution from Section 4.2, one can infer that it describes the object without border in a classical sense, therefore, its stability is supported not by surface tension but by nonlinear quantum effects in the bulk, similarly to the non-relativistic case [59]. Due to non-singular behaviour of the solution at the origin, the derivation of self-energy turned out to be entirely divergence-free.

The derived mass formula (41) suggested that for an electrically charged massive elementary particle there exists not only an antiparticle of the same mass (in the leading-order approximation with respect to the Planck constant, at least) but also a neutral particle of related mass. This might explain, at least qualitatively, why an electrically charged elementary particle is very often accompanied by a single neutral particle of a similar kind, but not *vice versa*. Indeed, such “mass pairing” feature has been observed (provided one disregards the influence of internal degrees of freedom such as spin, isospin, *etc.*) not only for elementary particles such as leptons and weak bosons, but also for stable composite ones such as nucleons (quarks might not fit this scheme since they are confined inside hadrons). We presented some arguments for why the rest mass of a neutral partner can sometimes be so much smaller, yet still non-vanishing, than the mass of the charged one (leptons), and sometimes they are of the same order of magnitude (weak bosons or nucleons).

Finally, we have compared the logarithmic vacuum model with one based on a Higgs-type (inverted quartic) potential. It turns out that the corresponding regular solution is unstable and energetically unfavourable, in contrast with the logarithmic case.

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## Appendix: Derivation of approximate solution

Here we provide more details regarding derivation of the approximate analytical solution from Section 4.2. Assuming (32) and (33), we will look for a solution of the system (16) and (17) in series form

$$\begin{aligned}\tilde{\phi} &= \tilde{\Phi}_0(x) + \xi \tilde{\Phi}_1(x) + \mathcal{O}(\xi^2), \\ \tilde{\psi} &= \tilde{\Psi}_0(x) \left( 1 + \xi \tilde{\Psi}_1(x) + \mathcal{O}(\xi^2) \right),\end{aligned}\quad (\text{A.1})$$

where  $\xi = \tilde{g}^2$ , and  $\tilde{\Phi}_i(x)$  and  $\tilde{\Psi}_k(x)$  are functions to be determined. Keeping terms of the order  $\mathcal{O}(\xi)$  and below, equations (16) and (17) can be reduced to the following set of four differential equations:

$$\tilde{\Phi}_0'' + \frac{2}{x} \tilde{\Phi}_0' = 0, \quad (\text{A.2})$$

$$\tilde{\Psi}_0'' + \frac{2}{x} \tilde{\Psi}_0' + \left[ \left( \tilde{E} + \tilde{\Phi}_0 \right)^2 + \ln(\tilde{\Psi}_0^2) \right] \tilde{\Psi}_0 = 0, \quad (\text{A.3})$$

$$\tilde{\Phi}_1'' + \frac{2}{x} \tilde{\Phi}_1' - 2 \left( \tilde{E} + \tilde{\Phi}_0 \right) \tilde{\Psi}_0^2 = 0, \quad (\text{A.4})$$

$$\begin{aligned}\tilde{\Psi}_1'' + 2 \left( \frac{1}{x} + \frac{\tilde{\Psi}_0'}{\tilde{\Psi}_0} \right) \tilde{\Psi}_1' + \\ + 2(\tilde{E} + \tilde{\Phi}_0) \tilde{\Phi}_1 + 2\tilde{\Psi}_1 = 0.\end{aligned}\quad (\text{A.5})$$

By solving them we obtain

$$\begin{aligned}\tilde{\phi} &= \Lambda + \xi \left( c_2 - \frac{c_1}{x} \right) - \\ &\quad - \frac{\xi \sqrt{\pi}}{2x} (\tilde{E} + \Lambda) e^{3-(\tilde{E}+\Lambda)^2} \text{erf}(x) + \mathcal{O}(\xi^2),\end{aligned}\quad (\text{A.6})$$

$$\tilde{\Psi}_0 = e^{(3-(\tilde{E}+\Lambda)^2-x^2)/2}, \quad (\text{A.7})$$

$$\begin{aligned}\tilde{\Psi}_1 &= \frac{1}{4} (\tilde{E} + \Lambda)^2 e^{3-(\tilde{E}+\Lambda)^2-x^2} \\ &\quad \left[ 1 + \frac{\sqrt{\pi}}{2x} (2x^2 + 1) e^{x^2} \text{erf}(x) \right] - \\ &\quad - c_2 (\tilde{E} + \Lambda) + c_3 \left( x - \frac{1}{2x} \right) + \\ &\quad + c_4 \left[ \frac{\sqrt{\pi}}{2x} (2x^2 - 1) \text{erfi}(x) - e^{x^2} \right],\end{aligned}\quad (\text{A.8})$$

where  $c_i$  and  $\Lambda$  are integration constants whose values must be fixed by means of the boundary conditions. Imposing (33), one obtains:  $c_1 = c_3 = c_4 = 0$  and  $c_2 = -\Lambda/\xi$ . Then, after making the energy redefinition

$\tilde{E} + \Lambda \rightarrow \tilde{E}$  (alternatively, one can set  $\Lambda = c_2 = 0$  from the beginning), one eventually arrives at expressions (34) and (35).

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